SOME ESSENTIALLY SELF-ADJOINT DIRAC OPERATORS WITH SPHERICALLY SYMMETRIC POTENTIALS†

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ABSTRACT

It is shown that the one electron Dirac operator in a stationary electric field is essentially self-adjoint, on the domain of infinitely differentiable functions of compact support, for a class of spherically symmetric potentials including the Coulomb potential, for atomic numbers less than or equal to 118. In addition, the domain of the closure of the perturbed operator is the same as the domain of the closure of the unperturbed operator. We also give an abstract theorem on domain-preserving essential self-adjointness for perturbed operators, which is perhaps of independent interest.

1. Introduction

It is known (Rellich [20], Weidmann [25]) that for e in the open interval $(-\sqrt{3}/2, \sqrt{3}/2)$ the Dirac operator

$$\dot{H}(e) = i^{-1}\alpha \cdot \operatorname{grad} + \beta + eV \text{ with } \mathfrak{D}(\dot{H}(e)) = \mathfrak{C}_0^{\infty,4}(R^3) \tag{1.1}$$

is essentially self-adjoint with $\mathfrak{D}(\overline{\dot{H}(e)}) = \mathfrak{D}(\overline{\dot{H}(0)}) = W_2^{1,4}(\mathscr{R}^3)$, in the case of the Coulomb potential $V(r) = -1/r$. In this paper we show that for e in the interval $(-\sqrt{3}/2, \sqrt{3}/2)$, domain-preserving essential self-adjointedness remains true for a class of spherically symmetric potentials including the Coulomb potential. Note that e is the adjusted atomic number defined by $e = aN$, where N is the atomic number and a is the fine structure constant, $a = \varepsilon^2(\hbar c)^{-1} \cong (137.037)^{-1}$. Hence the restriction $0 < e < \sqrt{3}/2$ corresponds to $N \leqq 118$, and

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(1.1) is the Dirac operator corresponding to an ion with one electron and atomic number N.

In Section 2 we define the operator of (1.1) in more detail. Theorem 2.1, which is our main theorem, says that if the potential V is spherically symmetric and dominated by the Coulomb potential, then for e in the interval $(-\sqrt{3}/2, \sqrt{3}/2)$ the Dirac operator $\dot{H}(e)$ is essentially self-adjoint with $\mathfrak{D}(\overline{\dot{H}(e)}) = W_2^{1,4}(\mathscr{R}^3)$. Our potential V need not possess any particular smoothness properties. In Section 3, we obtain an abstract theorem of Rellich-Kato type which gives necessary and sufficient conditions for preservation of (domain-preserving) essential self-adjointness under perturbations which are relatively bounded but with relative bound of arbitrary size. This Theorem 3.1 is perhaps of independent interest; hence in Section 3 we also observe some connections between this theorem and other perturbation criteria. In Section 4, we prove Theorem 2.1. This proof uses the fact that each one electron Dirac operator admits a complete family of reducing subspaces; since the orthogonal sum of essentially self-adjoint operators is essentially self-adjoint, it suffices to show that the part of $\dot{H}(e)$ in each of these subspaces is essentially self-adjoint. We show the latter by applying Theorem 3.1 to each of these parts. Roughly, the use of this theorem permits one to remove relatively compact parts of the perturbing operator; the key assumption of this theorem is the invertibility of an associated operator. We establish this invertibility with the aid of an estimate formulated elsewhere [19] combined with a result on invertibility of matrix operators (see Halmos [9, Problem 56]).

Let us make some brief additional comments on the history of this problem. By using the von Neumann and Weyl theories, it can be seen that (1.1) possesses a self-adjoint extension for all real values of e and a large class of real potentials V; for example, see L. Maurin [13]. In particular, this can be seen by showing that $\dot{H}(e)$ commutes with a conjugation. On the other hand, as concerns the essential self-adjointness which is desired if $\dot{H}(e)$ is to be a complete operator from the standpoint of quantum mechanics (restricting our attention for the moment to the Coulomb potential $V(r) = 1/r$), Case [4] observed that an additional boundary condition is needed for $N > 137$; later, Brownell [2] showed specifically that essential self-adjointness fails for e in $(1, \infty)$. Meanwhile, Kato [12] (see also [11]) observed that one obtains (domain-preserving) essential self-adjointness for e in $[0, \frac{1}{2}]$. Rellich [20] and Weidmann [25] extended essential self-adjointness to e in $[0, \sqrt{3}/2]$; Weidmann [25] also showed absolute con-

tinuity of the essential spectrum of $\overline{\dot{H}(e)}$. A limitation to values of N near 118 is in accordance with an earlier observation by Kato [12, p. 206] concerning domain changes for operators when represented in momentum space.

Prosser [17] showed essential self-adjointness for all e in the case of potentials V in $\mathfrak{L}^p$, $p > 3$, and Gross [6] showed that $\mathfrak{L}^3_{loc}$ is sufficient. Recently, Evans [5] extended Kato's result to V of the form $V(r) = V_0(r) + 1/r$, with e in $[0, \frac{1}{2}]$, and V_0 in the class of functions (see Stummel [24]) where $\int V_0^2(y)\,|x-y|^{-(1+\alpha)}dy$, $|x-y| \leqq 1$, is locally bounded for some fixed $0 < \alpha \leqq 1$. It was also shown in [5] by a constructive method that the Green's matrix exists for (1.1), for the Coulomb potential and for e in $[\sqrt{3}/2)$; although not explicitly stated there, the essential self-adjointness for e in $[0, \sqrt{3}/2)$ follows from [5, Theorem 2] and essential self-adjointness extends to the potentials $V = V_0 + 1/r$ mentioned above for e in $[0, \sqrt{3}/2)$ in the same manner as was done for the smaller interval in [5]. Jorgens [10] has very recently obtained similar general results for matrix valued potentials $V = A_1 + A_2 + A_3$, with $|A_1(x)|$ in $\mathfrak{L}^3_{loc}(\mathscr{R}^3)$, $|A_2(x)|$ in the Stummel class mentioned above, and $A_3(x)$ the sum of Coulomb potentials $|x-x_k|^{-1}B_k$ with B_k constant matrices such that either $|B_k| < \frac{1}{2}$ or such that $B_k = e_k I$ with $|e_k| < \sqrt{3}/2$; here $|A(x)|$ denotes the maximum of the absolute values of $A(x)$. Jorgens [10] also contains nonlocal versions. We also mention the work of Schillemeit [22] and the paper [1] for a group-theoretic approach to this problem.†

2. The Dirac operator and the essential self-adjointness theorem

The operator $\dot{H}(e)$ in (1.1) acts on vectors $u(x) = (u_1(x), \ldots, u_4(x))$ with four components, where x is the space variable $x = (x_1, x_2, x_3)$. Thus the basic Hilbert space is $\mathfrak{H} = \oplus \sum_{n=1}^{4} \mathfrak{L}^2_n(\mathscr{R}^3)$, with inner product $(u, v) = \sum_{n=1}^{4} (u_n, v_n)_n$; we shall use the notation $\mathfrak{H} = [\mathfrak{L}^2(\mathscr{R}^3)]^4$, e.g., see [10, 12]. The α in (1.1) has three components $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ with each component α_k a 4×4 matrix; similarly, β is a 4×4 matrix, and these Dirac matrices α_k and β satisfy the commutation relations $[\alpha_k, \alpha_l] = 2\delta_{kl}$, here letting $\alpha_4 = \beta$ for convenience. The potential V in (1.1) represents the operator of multiplication of each component of u by a real function $V = V(x)$. Thus $\dot{H}(e)u = v \in \mathfrak{H}$ has first component

$$v_1(x) = i^{-1} \sum_{k=1}^{3} \sum_{h=1}^{4} (\alpha_k)_{1h} \frac{\partial u_h(x)}{\partial x_k} + \sum_{h=1}^{4} (\beta)_{1h} u_h(x) + eV(x)u_1(x), \tag{2.1}$$

† *Added in proof:* We have just learned about the paper of Schmincke [23].

the other three components being similar. We refer the reader to [11, 2, 5, 19, 14] for further details.

It is known (e.g., see [5, 14]) that the Dirac operator for the Coulomb potential can be treated by separation of variables. Let V be spherically symmetric and dominated by the Coulomb potential; that is, we assume that V is of the form

$$V(x) = p(|x|) \tag{2.2}$$

for some real valued function p defined on $(0, \infty)$ such that

$$p(|x|) \leqq |x|^{-1}, \quad |x| \neq 0. \tag{2.3}$$

Then $\dot{H}(e)$ is, by separation of variables, unitarily equivalent to a system of 2×2 matrix ordinary differential operators (see Section 4) on a complete family of reducing (in the sense of [11, pp. 171–172]) subspaces for $\dot{H}(e)$, so that it suffices to demonstrate essential self-adjointness for the part $\dot{H}(e)$ in each reducing subspace. As will be seen in Section 4, the estimates used to do the latter depend only on the growth rate of the potential and hence we do not assume any smoothness or special properties in (2.3).

We describe a specific domain of essential self-adjointness as follows. Let T_n, $n = 1, \ldots, 4$, be the operator taking $\mathfrak{L}^2_n(\mathscr{R}^3)$ onto $(\mathfrak{L}^2_n(\mathscr{S}_2), \mathfrak{L}^2_n(0, \infty))$ that introduces spherical coordinates, that is, $(T_n f)(r, \theta, \phi) = rf(x_1(r, \theta, \phi), x_2(r, \theta, \phi), x_3(r, \theta, \phi))$, where $\mathscr{S}_2$ denotes the two-dimensional unit sphere, and let T be the operator on $[\mathfrak{L}^2(\mathscr{R}^3)]^4$ given by applying T_n on each component. Then $\dot{H}(e)$ is unitarily equivalent to $T\dot{H}(e)T^{-1}$ and $\mathfrak{D}(T\dot{H}(e)T^{-1}) = [\mathfrak{C}^\infty(\mathscr{S}_2), \mathfrak{C}^\infty_0(0, \infty)]^4$ if the domain of $\dot{H}(e)$ is taken to be

$$\dot{\mathfrak{D}} = T^{-1}\{[\mathfrak{C}^\infty(\mathscr{S}_2), \mathfrak{C}^\infty_0(0, \infty)]^4\}. \tag{2.4}$$

The proof of the following main theorem will be given in Section 4, based on Theorem 3.1 of the next section.

THEOREM 2.1. *Let the operator $\dot{H}(e)$ be defined as in* (1.1) *with $\mathfrak{D}(\dot{H}(e)) \equiv \dot{\mathfrak{D}}$ as in* (2.4) *and with a potential V that is spherically symmetric and dominated by the Coulomb potential as in* (2.2) *and* (2.3). *Then $\dot{H}(e)$ is essentially self-adjoint for e in $(-\sqrt{3}/2, \sqrt{3}/2)$, and $\mathfrak{D}(\overline{\dot{H}(e)}) = \mathfrak{D}(\overline{\dot{H}(0)}) = W_2^{1,4}(\mathscr{R}^3)$.*

We remark that since $\dot{\mathfrak{D}} \subset [\mathfrak{C}^\infty_0(R^3)]^4$, essential self-adjointedness will hold on the latter domain also.

3. An abstract theorem on essential self-adjointness

As is well-known (e.g., see Kato [11, p. 272]), a symmetric operator T is essentially self-adjoint if and only if the ranges $\mathfrak{R}(T-\mu_{\pm})$ are dense, for some $\mu_{\pm}$ with $\operatorname{Im}\mu_{+}>0$, and some μ_{-} with $\operatorname{Im}\mu_{-}<0$. Rellich and Kato formulated criteria on a perturbation which ensured that the perturbed and unperturbed operators are essentially self-adjoint on the same domain. Specifically, the Rellich-Kato Theorem (e.g., see [11, p. 288]) states that if A_0 is essentially self-adjoint, V is symmetric with $\mathfrak{D}(V)\supset\mathfrak{D}(A_0)$, and V is A_0-bounded with relative bound strictly less than 1, i.e., there exist constants a and b, $b<1$, such that

$$\|Vx\| \leqq a\|x\| + b\|A_0x\|, \quad x\in\mathfrak{D}(A_0), \tag{3.1}$$

then A_0+V is essentially self-adjoint and

$$\overline{A_0+V} = \bar{A}_0+\bar{V} \text{ and } \mathfrak{D}(\overline{A_0+V}) = \mathfrak{D}(\bar{A}_0). \tag{3.2}$$

In this section, we replace the sufficient condition of relative bound less than 1 by another condition, formulated in terms of the Fredholm index of associated operators, which is necessary and sufficient in the class of perturbations V which are A_0-bounded, i.e., V for which there exists some constant c such that

$$\|Vx\| \leqq c[\|x\| + \|A_0x\|], \quad x\in\mathfrak{D}(A_0). \tag{3.3}$$

This extended version of the Rellich-Kato Theorem is useful, for example, as in Section 4 of this paper, when treating perturbations V which are perhaps not small perturbations. The Fredholm terminology, although not essential, conveniently isolates the principal requirements on V relative to A_0.

Concerning the condition (3.3), let us recall some elementary facts in the following lemma.

LEMMA 3.1. *Let $\mathfrak{D}(V)\supset\mathfrak{D}(A_0)$, where V and A_0 are both closeable operators in a Banach space $\mathfrak{X}$. Then equivalently:*

(i) *V is A_0-bounded,* i.e., (3.3) *holds,*

(ii) *$\bar{V}$ is $\bar{A}_0$-bounded,*

(iii) *$V(\mu-A_0)^{-1}$ is bounded, for every μ in the resolvent set of A_0,*

(iv) *$\bar{V}(\mu-\bar{A}_0)^{-1}\in\mathfrak{B}(\mathfrak{X})$, for every μ in the resolvent set of $\bar{A}_0$,*

(v) *$\mathfrak{D}(\bar{A}_0+\bar{V}) = \mathfrak{D}(\bar{A}_0)$,*

(vi) *$\mathfrak{D}(\bar{V})\supset\mathfrak{D}(\bar{A}_0)$.*

For such V and A, one always has $\overline{A_0+V}\supset\bar{A}_0+\bar{V}$ if A_0+V is closeable, with equality if and only if $\bar{A}_0+\bar{V}$ is closed.

The verification of Lemma 3.1 follows from the closed graph theorem and elementary manipulations, e.g., as in [11, p. 288]. We recall that for non-closed A_0, the resolvent set is defined to be those μ for which $(\mu - A_0)^{-1}$ is bounded and densely defined; for the equivalences (iii) and (iv) we tacitly assume that the resolvent sets there are not empty. We would like to mention, even though it is obvious, that the condition (vi), that is,

$$\mathfrak{D}(\bar{V}) \supset \mathfrak{D}(\bar{A}_0) \tag{3.4}$$

which is equivalent to relative boundedness, is often easy to check in specific applications, and, for example, is immediate when it is known that $\mathfrak{D}(V) \supset \mathfrak{D}(\bar{A}_0)$. This, then, is the restriction on the class of potentials that can be considered in Theorem 3.1 below.

Since our viewpoint is Fredholm theory, we recall that a densely defined operator in a Banach space $\mathfrak{X}$ is said to be Fredholm if T is closed, $\mathfrak{R}(T)$ is closed, and both $\alpha(T)$ and $\beta(T)$ are finite, where $\alpha(T) = \dim \mathfrak{N}(T)$ and $\beta(T) = \dim \mathfrak{X}/\mathfrak{R}(T)$; T is Fredholm of index zero if in addition the index $i(T) = 0$, where $i(T) = \alpha(T) - \beta(T)$.

THEOREM 3.1. *Let A_0 be essentially self-adjoint, V symmetric with $\mathfrak{D}(V) \supset \mathfrak{D}(A_0)$, where V is A_0-bounded. For each μ in the resolvent set of A_0 define the operator $T_\mu \in \mathfrak{B}(\mathfrak{H})$ by*

$$T_\mu = I - \bar{V}(\mu - \bar{A}_0)^{-1}.$$

Then $A_0 + V$ is essentially self-adjoint, $\overline{A_0 + V} = \bar{A}_0 + \bar{V}$, and $\mathfrak{D}(\overline{A_0 + V}) = \mathfrak{D}(\bar{A}_0)$, if and only if there exist μ_+ in the closed upper half-plane and μ_- in the closed lower half plane such that the operators T_{μ_+} and T_{μ_-} are Fredholm of index zero.

PROOF. Using the identity

$$\mu - \bar{A}_0 - \bar{V} = [I - \bar{V}(\mu - \bar{A}_0)^{-1}](\mu - \bar{A}_0), \tag{3.5}$$

and since $\mu - \bar{A}_0$ is Fredholm of index zero for all μ in the resolvent set of $\bar{A}_0$, the sufficiency follows from the well-known fact (e.g., Schechter [21, Theorem VII.1.3]) that the composition of two Fredholm operations A_1 and A_2 is Fredholm and the index of the composition is the sum of the indices of the two operators, that is, $i(A_1 A_2) = i(A_1) + i(A_2)$. Thus, for complex μ_+ and μ_-, the defect indices $\beta(\mu_+ - \bar{A}_0 - \bar{V}) = \beta(\mu_- + \bar{A}_0 - \bar{V}) = 0$ and $\bar{A}_0 + \bar{V}$ is self-adjoint, whereas for real $\mu = \mu_+ = \mu_-$, one has an operator $\mu - \bar{A}_0 - \bar{V}$ which is sym-

metric and onto, hence self-adjoint, so that $\bar{A}_0 + \bar{V}$ is self-adjoint. The necessity follows for example from the fact (see Schechter [21, Theorem VII. 2.5]) that if the composition A_1A_2 is Fredholm, where A_2 is Fredholm and A_1 is closed, then A_1 is Fredholm, so that by the index formula mentioned above applied to (3.5), we have $T_{\pm i}$ is Fredholm of index zero.

In Section 4, we will use Theorem 3.1 in the following form, there taking $\mu_+ = \mu_- = 0$.

COROLLARY 3.1 [19, Theorem 3.1] *If there exist μ_+ and μ_- as in Theorem* 3.1 *such that $T_{\mu\pm} = B_\pm + C_\pm$, where $B_\pm$ are invertible in the sense $B_\pm^{-1} \in \mathfrak{B}(\mathfrak{H})$, and $C_\pm$ are compact, then $A_0 + V$ is essentially self-adjoint and $\mathfrak{D}(\overline{A_0 + V}) = \mathfrak{D}(\bar{A}_0)$.*

Corollary 3.1 follows from the fact that an invertible operator is Fredholm of index zero and that this property is stable under compact perturbations, (e. g., see [11, Theorem IV.5.17], [21, Theorem VII.2.1]).

For μ nonreal, T_μ is already 1-1, since $T_\mu u = 0$, $u \neq 0$, would imply that $(\bar{V}v, v) = \mu \| v \|^2 - (\bar{A}_0 v, v)$ for $v = (\mu - \bar{A}_0)^{-1}u$, requiring that μ be real by the symmetry of $\bar{A}_0$ and $\bar{V}$. If $\mu_+ = \mu_- = \mu$ real, then $\beta(T_\mu) = 0$ implies that the symmetric operator $\mu - \bar{A}_0 - \bar{V}$ has a dense range. Therefore one can state Theorem 3.1 in terms of the following weaker (but equivalent) hypothesis, showing that Theorem 3.1 has the effect of shifting the necessary and sufficient conditions for essential self-adjointness, mentioned at the beginning of this section, from the unbounded operators $\mu - A_0 - V$ to the bounded operators $T_{\mu\pm}$.

COROLLARY 3.2. *With A_0, V, and T_μ as in Theorem* 3.1, *$A_0 + V$ is essentially self-adjoint and $\mathfrak{D}(\overline{A_0 + V}) = \mathfrak{D}(\bar{A}_0)$ if and only if there exist $\mu_\pm$ as in Theorem* 3.1 *such that $\beta(T_{\mu\pm}) = 0$.*

Theorem 3.1 includes the Rellich-Kato Theorem. As in Kato [11, p. 288], noting that with $\mu_\pm = \pm ic'$, $c' = a'/b'$, with a' and b' as in [11], one has $\| \bar{V}(\mu_\pm - \bar{A}_0)^{-1} \| < 1$; hence the $T_{\mu\pm}$ are invertible, and therefore Fredholm of index zero. Similarly, one can check that the following modification of Theorem 3.1 includes the sufficient condition of relative bound $b' = 1$, e.g., see [11, p. 289–290]. To illustrate the difference between the conclusions, we remove the hypothesis of Theorem 3.1 that V be A_0-bounded.

COROLLARY 3.3. *Let A_0 be essentially self-adjoint, V symmetric with $\mathfrak{D}(V) \supset \mathfrak{D}(A_0)$, and define $T_\mu = I - V(\mu - A_0)^{-1}$ for μ in the resolvent set of A_0. Then $A_0 + V$ is essentially self-adjoint if and only if there exist $\mu_\pm$ in the open upper and lower half-planes such that the ranges $\mathfrak{R}(T_{\mu\pm})$ are dense.*

PROOF. Both the necessity and sufficiency follow from (3.5) written with A_0 and V in place of $\bar{A}_0$ and $\bar{V}$, the necessity with $\mu_\pm = \pm i$, the sufficiency with the postulated $\mu_\pm$.

Theorem 3.1 has some connections with other perturbation criteria for essential self-adjointness and contraction semigroups obtained by "doubling" or other continuity arguments (e.g., see [3, 7, 8, 16]). Without further elaboration, we illustrate this point by extending the result of Okazawa [16, Corollary], namely, that if A is self-adjoint, V symmetric, $\mathfrak{D}(V) \supset \mathfrak{D}(A)$, and $\mathrm{Re}(Au, Vu) \geqq 0$ for all $u \in \mathfrak{D}(A)$, then $A + V$ is self-adjoint.

COROLLARY 3.4. *Let A_0 be essentially self-adjoint, V symmetric with $\mathfrak{D}(V) \supset \mathfrak{D}(A_0)$, and V A_0-bounded. Then $A_0 + V$ is essentially self-adjoint and $\overline{A_0 + V} = \bar{A}_0 + \bar{V}$ if the set of values (A_0u, Vu), for all $u \in \mathfrak{D}(A_0)$, is contained in some closed half-plane not containing $(-\infty, 0]$.*

PROOF. Defining T_μ as in Theorem 3.1 or Corollary 3.3, we first observe that since T_μ is bounded and densely defined, the numerical ranges $\overline{W(T_\mu)}$ and $\overline{W(T^*_\mu)}$ are conjugates of each other; recall that for a closeable operator T, one has $\overline{W(\bar{T})} = \overline{W(T)}$ by the joint continuity of the inner product. Hence if $0 \notin \overline{W(T_{\mu_\pm})}$ for some acceptable $\mu_\pm$ in the open upper and lower half-planes, $T_{\mu_\pm}$ and $T^*_{\mu_\pm}$ will all have bounded inverses (hence dense ranges) so that $A_0 + V$ will be essentially self-adjoint by Corollary 3.3 and $\overline{A_0 + V} = \bar{A}_0 + \bar{V}$ by Theorem 3.1.

For $\|x\| = 1$, $x = (\mu - A_0)y$, $|\lambda| = 1$, we have

$$(\lambda T_{\bar{\mu}} x, x) = \lambda - \lambda\mu(Vy, y) + \lambda(Vy, A_0 y). \tag{3.6}$$

Now choose $\lambda = e^{i\theta}$, $|\theta| < \pi/2$, such that $\lambda(Vy, A_0y)$ is rotated to have a nonnegative real part, and then let $\mu_\pm = \pm i\bar{\lambda}$. Then

$$\begin{aligned} \mathrm{Re}(\lambda T_{\bar{\mu}_\pm} x, x) &= \mathrm{Re}\,\lambda - \mathrm{Re}\{i|\lambda|^2(Vy, y)\} + \mathrm{Re}\{\lambda(Vy, A_0y)\} \\ &\geqq \mathrm{Re}\,\lambda, \end{aligned} \tag{3.7}$$

so that the numerical ranges $\overline{W(T_{\mu_\pm})}$ exclude zero.

We mention that the half-plane condition on (A_0u, Vu) on $\mathfrak{D}(A_0)$ is equivalent to the same condition $(\bar{A}_0u, \bar{V}u)$ by the continuity of the inner product and the A_0-boundedness of V.

4. Application to the Dirac operator

By assumption (2.2), the potential V is spherically symmetric. This implies that the operator $T\dot{H}(e)T^{-1}$, defined in Section 2, admits a complete family of

reducing subspaces and on each of them it is unitarily equivalent to a matrix differential operator

$$L(e)(\kappa) = \begin{bmatrix} (I - ep(\xi)) & \left(\frac{-d}{d\xi} + \frac{\kappa}{\xi}\right) \\ \left(\frac{d}{d\xi} + \frac{\kappa}{\xi}\right) & -(I + ep(\xi)) \end{bmatrix} \tag{4.1}$$

$$\mathfrak{D}[L(e)(\kappa)] = [\mathfrak{C}_0^\infty(0, \infty)]^2$$

$$\kappa = \pm 1, \pm 2, \pm 3, \ldots .$$

This is known for the Coulomb potential (see [18],[14]) and follows for the V considered here as in [18], with straightforward adjustments. Since $T\dot{H}(e)T^{-1}$ is unitarily equivalent to $\dot{H}(e)$, and since the orthogonal sum of essentially self-adjoint operators is essentially self-adjoint, the main Theorem 2.1 is implied by the following theorem.

THEOREM 4.1. *Let κ be fixed and let e be in $(-\sqrt{3}/2, \sqrt{3}/2)$. Then the operator $L(e)(\kappa)$ is essentially self-adjoint in $[\mathfrak{L}^2(0, \infty)]^2$.*

PROOF. For each κ we apply the Corollary 3.1 version of Theorem 3.1, with

$$A_0 = L(0)(\kappa), \; V = M(p), \; \mu_+ = \mu_- = 0, \tag{4.2}$$

where V is the operator given by multiplication of each of the two components by the function p. The A_0-boundedness of V follows from assumption (2.3) and from a well-known Sobolev estimate (e.g., see [11, p. 307] or [19, Lemma 3.2]). In the lemma that follows we show that the other assumptions of Corollary 3.1 also hold. We first recall (e.g., see Prosser [17]) that $\dot{H}(0)$ is essentially self-adjoint on $\dot{\mathfrak{D}}$ and hence each $L(0)(\kappa)$ is essentially self-adjoint on its domain, and also that $\mu = 0$ is in the resolvent set of $\dot{H}(0)$ and hence in the resolvent set of $L(0)(\kappa)$, so that we may take $\mu_+ = \mu_- = 0$. In order to maintain the connection between our notation here and that of [19] so that we may use some estimates from the latter, we also designate the operator A_0^{-1} by

$$A_0^{-1} \equiv R_0(\kappa)(0).$$

LEMMA 4.1. *For each fixed κ, for the operator $R_0(\kappa)(0)$ there is a family of operators $\{A_\delta(\kappa)\}$ such that for each positive δ*

$$M(p)R_0(\kappa)(0) - A_\delta(\kappa) \equiv C_\delta \text{ is compact}. \tag{4.3}$$

Moreover for each number e in the open interval $(-\sqrt{3}/2, \sqrt{3}/2)$ *the number* δ *can be chosen so that*

$$(I - eA_\delta(\kappa))^{-1} \in \mathfrak{B}\{[\mathfrak{L}_2(0,\infty)]^2\}. \tag{4.4}$$

In other words, in terms of Theorem 3.1 and Corollary 3.1, letting

$$T_0 \equiv T_\pm = I - VA_0^{-1},\ B_0 = I - eA_\delta,\ \text{and}\ C_0 = C_\delta,$$

we have

$$T_0 = B_0 + C_0$$

with B_0 and C_0 as required in Corollary 3.1.

The proof of Lemma 4.1 is based on the following (4.5) known representation of the kernel $R_0(\kappa)(0)(\xi,\eta)$ of the operator $R_0(\kappa)(0)$ (e.g., see [19, Lemma 3.1] or [5]). For brevity we consider only the case of positive values of κ, the case for negative κ being similar. One has

$$R_0(\kappa)(0)(\xi,\eta) = \begin{cases} F(\xi,\eta), & \eta < \xi, \\ F^*(\xi,\eta), & \xi < \eta, \end{cases} \tag{4.5}$$

where

$$F(\xi,\eta) = \frac{\pi}{2i}(\xi\eta)^{\frac{1}{2}} \begin{bmatrix} H^{(1)}_{\alpha_1(\kappa)}(i\xi)J_{\alpha_1(\kappa)}(i\eta), & -iH^{(1)}_{\alpha_1(\kappa)}(i\xi)J_{\alpha_2(\kappa)}(i\eta) \\ iH^{(1)}_{\alpha_2(\kappa)}(i\xi)J_{\alpha_1(\kappa)}(i\eta), & -H^{(1)}_{\alpha_2(\kappa)}(i\xi)J_{\alpha_2(\kappa)}(i\eta) \end{bmatrix} \tag{4.6}$$

and where

$$\alpha_1(\kappa) = \kappa + \tfrac{1}{2} \text{ and } \alpha_2(\kappa) = \kappa - \tfrac{1}{2}. \tag{4.7}$$

Here, of course, J and H are the Bessel and Hankel functions in the usual notations.

To verify conclusion (4.3), we define the kernel of the operator $A_\delta(\kappa)$ to be the singular part of the kernel of $M(p)R_0(\kappa)(0)$ near the origin. Specifically, let c_δ denote the characteristic function of the interval $[0,\delta]$ and set

$$A_\delta(\kappa)(\xi,\eta) = c_\delta(\xi)c_\delta(\eta) \begin{cases} G(\kappa)(\xi,\eta), & \eta < \xi, \\ G^*(\kappa)(\xi,\eta), & \xi < \eta, \end{cases} \tag{4.8}$$

where

$$G(\kappa)(\xi,\eta) = -\frac{\pi}{2}(\xi,\eta)^{\frac{1}{2}} \begin{bmatrix} 0 & p(\xi)H^{(1)}_{\alpha_1(\kappa)}(i\xi)J_{\alpha_2(\kappa)}(i\eta) \\ 0 & 0 \end{bmatrix} \tag{4.9}$$

For the case of the Coulomb potential, that is, for $p(\xi) = 1/\xi$, conclusion (4.3) was verified elsewhere [19, conclusion (3.4) of Theorem 3.2]. Those arguments involved only the absolute value of the kernel of the operator in (4.3). Assumption (2.3) shows that the absolute value of this kernel is majorized by the absolute value of the kernel corresponding to the Coulomb potential. Clearly if the absolute value of the kernel of an integral-operator admits a majorant kernel such that the corresponding operator is compact, then so is the original operator. These facts together establish the validity of conclusion (4.3) under the present assumption (2.3).

To establish conclusion (4.4), let the projectors P_1 and P_2 acting on $[\mathfrak{L}^2(0,\infty)]^2$ be defined by

$$P_1\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ 0 \end{bmatrix} \text{ and } \quad P_2\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} 0 \\ f_2 \end{bmatrix}.$$

Then it follows from definitions (4.8) and (4.9) that

$$P_1 A_\delta(\kappa) P_1 = P_2 A_\delta(\kappa) P_2 = 0. \tag{4.10}$$

According to an estimate formulated elsewhere [19, estimates $(3.56)_{12}$ and $(3.56)_{21}$], to each κ and positive number ε there is a strictly positive number δ such that

$$\| P_1 A_\delta(\kappa) P_2 \| \leqq \frac{1+\varepsilon}{\alpha_1(\kappa)} \text{ and } \| P_2 A_\delta(\kappa) P_1 \| \leqq \frac{1+\varepsilon}{\alpha_2(\kappa)}. \tag{4.11}$$

This was stated for in [9] the case of the Coulomb potential, but assumption (2.3) allows us to conclude it for our present class of potentials. Inserting definition (4.7) in these estimates, we obtain

$$\| P_1 A_\delta(\kappa) P_2 \| \cdot \| P_2 A_\delta(\kappa) P_1 \| \leqq \frac{(1+\varepsilon)^2}{\kappa^2 - 1/4}, \tag{4.12}$$

if we remember that according to definition (4.7)

$$\alpha_1(\kappa)\alpha_2(\kappa) = \kappa^2 - 1/4, \ \kappa = 1, 2, \cdots.$$

Hence

$$e^2(1+\varepsilon)^2 < \kappa^2 - 1/4 \tag{4.13}$$

implies

$$e^2 \| P_1 A_\delta(\kappa) P_2 A_\delta(\kappa) P_1 \| < 1. \tag{4.14}$$

Thus for e in $(-\sqrt{3}/2, \sqrt{3}/2)$ we can choose ε in (4.13) and a corresponding

δ in (4.11) so that (4.14) holds. The relations (4.14) and (4.10) are sufficient to imply the conclusion (4.4); e.g., apply [9, Problem 56] to the operator

$$\begin{bmatrix} I & -eP_1A_\delta(\kappa)P_2 \\ -eP_2A_\delta(\kappa)P_1 & I \end{bmatrix}.$$

We note that in this way we have also extended the interval size beyond that considered in [19]. Roughly speaking, we have used the spectral radius of $eA_\delta(\kappa)$ rather than its numerical radius to determine the invertibility of $I - eA_\delta(\kappa)$. This is in accordance with the well-known fact that the numerical radius majorizes the spectral radius (e.g. see [9, Problem 173]).

This completes the proof of Lemma 4.1, hence of Theorem 4.1, and hence of Theorem 2.1.

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